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THE GAUSSIAN MINIMUM ENTROPY CONJECTURE & ITS IMPORTANCE FOR THE INFORMATION CAPACITY OF GAUSSIAN BOSONIC CHANNELS

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Fundamentals, and Applications”

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Context

We wish to calculate the **ultimate** information capacity of optical communication channels



- Since communication channels are **“physical devices”** ...
- Since physics itself is **“quantum mechanical”** ...

... we need to use the tools of **“quantum information theory”**

Outline

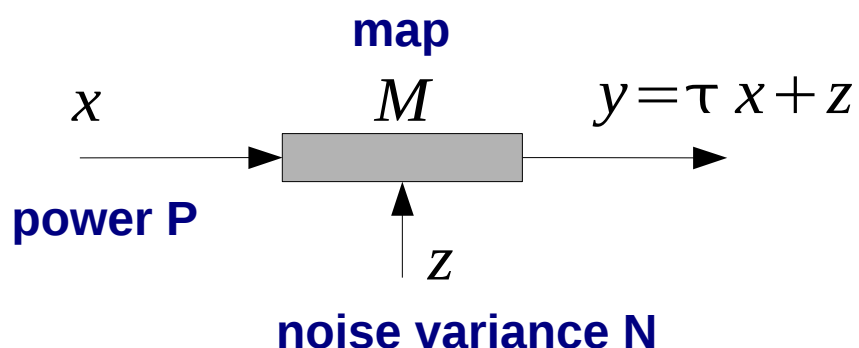
- Gaussian bosonic channels
Classical capacity of quantum channels
Gaussian minimum (output) entropy conjecture
- Link with entanglement of a 2-mode squeezer
Gaussian minimum (output) entanglement conjecture
- Connection with majorization theory
(Stronger) Gaussian majorization conjecture
Incomplete proof of the conjecture (Fock state inputs)
- Conclusions — importance for physics problems !
... new path to the ultimate proof ?

Motivation: perfect (noiseless) channel



Shannon theory

Numerous communication links modeled by a classical **Gaussian additive-noise channel**



$$C(M) = \frac{1}{2} \log \left(1 + \frac{\tau P}{N} \right)$$

$$\begin{aligned} \tau &= 1 \\ N &= 0 \end{aligned}$$

Infinite capacity for finite input power P !

..... we need quantum mechanics to calculate the ultimate limits of communication !

Quantum theory

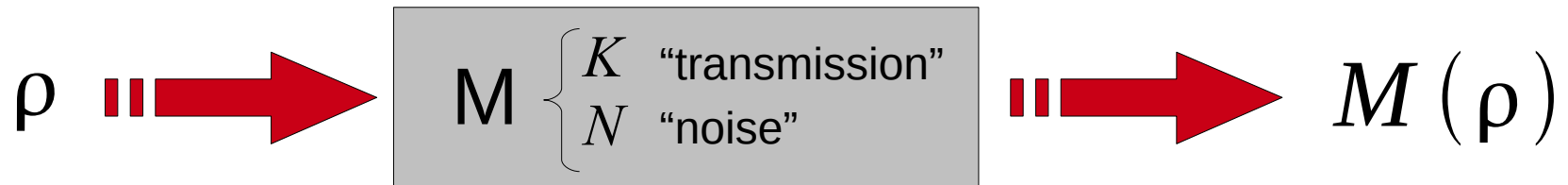
Gaussian quantum channels

H. P. Yuen, and M. Ozawa, PRL 1993

$$C(M) = \max_{\rho} S(\rho) = (\nu + 1) \log(\nu + 1) - \nu \log(\nu) \quad \text{for identity channel}$$

Finite capacity for finite input energy, achieved by a thermal state of ν photons

Gaussian (quantum) Bosonic channels



- Corresponds to linear CP maps $\rho \rightarrow M[\rho]$
s.t. $M[\rho]$ Gaussian if ρ Gaussian

- M fully characterized by two matrices K, N

$$\vec{r} \rightarrow K \vec{r} \quad \vec{r} = \text{coherent vector}$$

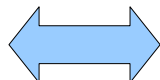
$$\gamma \rightarrow K \gamma K^T + N \quad \gamma = \text{covariance matrix}$$

↑
real

↑
real & symmetric

one-mode case

- M completely positive

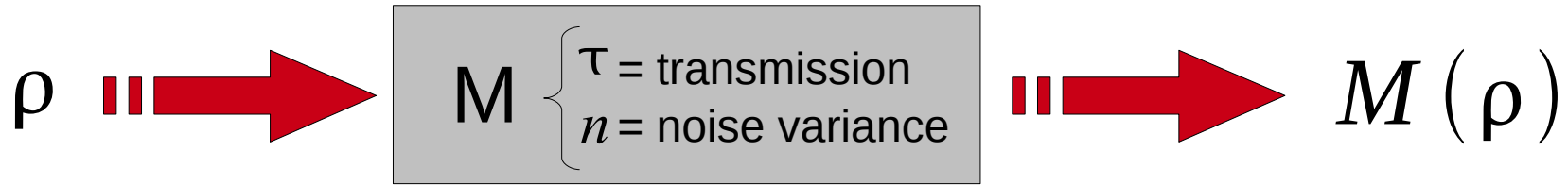


$$N \geq 0 \quad \det N \geq (\det K - 1)^2$$

uncertainty principle

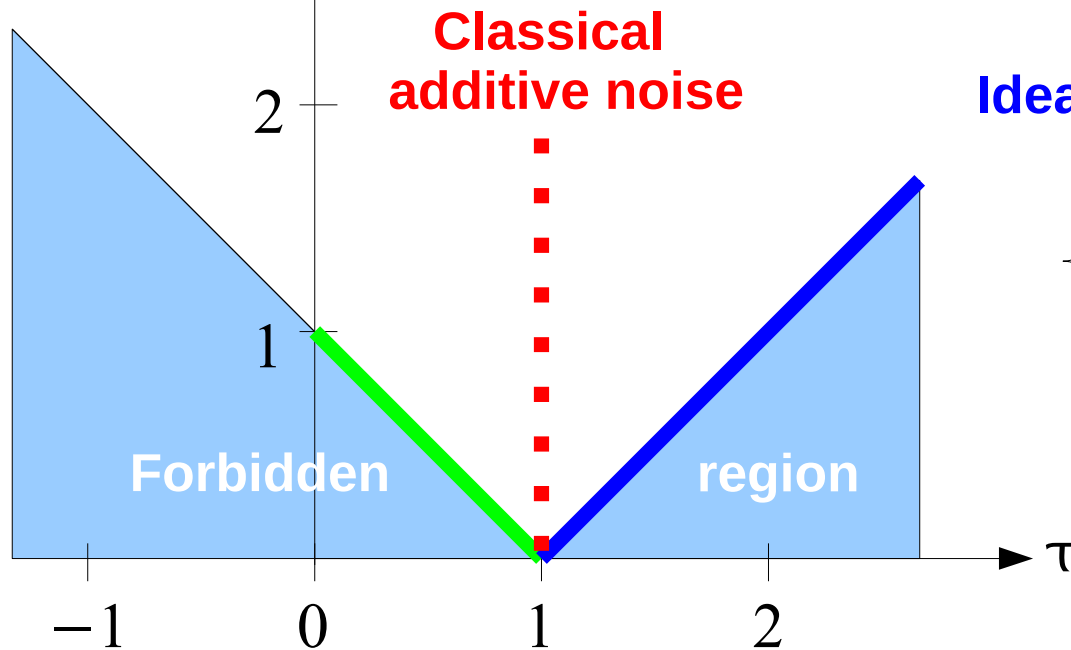
Phase-insensitive Gaussian Channels

$$\begin{cases} K = \text{diag}(\sqrt{\tau}, \sqrt{\tau}) \\ N = \text{diag}(n, n) \end{cases}$$



Purely lossy channel

$$\begin{cases} K = \text{diag}(\sqrt{T}, \sqrt{T}) & (T \leq 1) \\ N = \text{diag}(1-T, 1-T) \end{cases}$$



Ideal (quantum-limited) amplifier

$$\begin{cases} K = \text{diag}(\sqrt{G}, \sqrt{G}) & (G \geq 1) \\ N = \text{diag}(G-1, G-1) \end{cases}$$

(our analysis is focused on phase-insensitive channels w.n.l.g. 😊)

Classical Capacity of Quantum Channels

Holevo, Schumacher,
Westmoreland, 1998

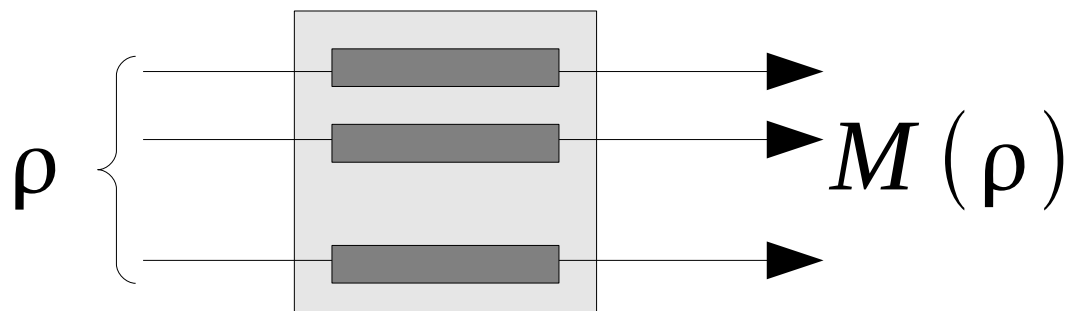


encoding $\{p_a, \rho_a\}$ such that $\sum_{a=1, \dots, d} p_a \rho_a = \rho$

- **Holevo bound** $\chi(\{p_a, \rho_a\}, M) = S(M(\rho)) - \sum_a p_a S(M(\rho_a))$
- **Single-shot capacity** $C^{(1)}(M) = \max_{\{p_a, \rho_a\}} \chi(\{p_a, \rho_a\}, M)$
- **Capacity** $C(M) = \lim_{n \rightarrow \infty} \frac{1}{n} C^{(1)}(M^{\otimes n})$

... in general, not additive !

(Hastings, Nature Phys. 2009)



Minimum Output Entropy



encoding $\{p_a, \rho_a\}$ such that $\sum_{a=1, \dots, d} p_a \rho_a = \rho$

$$C^{(1)}(M) = \max_{\rho} \tilde{\chi}(\rho, M) \quad \dots \text{maximization in 2 steps}$$

$$\text{with } \tilde{\chi}(\rho, M) \equiv S(M(\rho)) - \min_{\substack{\{p_a, \rho_a\} \\ \sum_a p_a \rho_a = \rho}} \underbrace{\sum_a p_a S(M(\rho_a))}_{\geq S(M(\Phi_0))}$$

$$\leq S(M(\rho)) - S(M(\Phi_0))$$

→ we need to find pure state Φ_0 minimizing output entropy

$$\min_{\sigma} S(M(\sigma)) \equiv S(M(\Phi_0))$$

Capacity of Gaussian Quantum Channels

Yuen and Ozawa, 1993
Holevo and Werner, 1998

- continuous encoding
- energy constraint



encoding $\{p(\alpha), \rho_\alpha\}$ such that $\int d^2\alpha p(\alpha) \rho_\alpha = \rho$

$$C^{(1)}(M) = \max_{\rho} \tilde{\chi}(\rho, M)$$

$$\leq \max_{\rho} S(M(\rho)) - S(M(\Phi_0))$$

for fixed energy,
achieved by a thermal state

$$S(M(\rho_{therm}))$$

minimum
output entropy
state ???

Gaussian minimum output entropy conjecture:

$$\Phi_0 = |0\rangle\langle 0|$$

Conjectured Single-shot Capacity

$$\rho_\alpha \xrightarrow{\text{ }} \boxed{M \begin{array}{l} \tau = \text{transmission} \\ n = \text{noise variance} \end{array}} \xrightarrow{\text{ }} M(\rho_a)$$

encoding $\{p(\alpha), \rho_\alpha\}$ such that $\int d^2\alpha p(\alpha) \rho_\alpha = \rho$

$$C^{(1)}(M) \leq S(M(\rho_{therm})) - S(M(|0\rangle\langle 0|)) \xrightarrow{\text{ }} \Phi_0$$

★ use encoding $\rho_\alpha = D(\alpha)|0\rangle\langle 0|D(\alpha)^\dagger$ $p(\alpha) = \frac{1}{\sqrt{\pi\nu}} \exp\left(\frac{-|\alpha|^2}{\nu}\right)$

with ν = mean thermal photon number

$$C^{(1)}(M) = g[\tau\nu + n] - g[n] \quad \text{where } g[x] = (x+1)\log(x+1) - x\log(x)$$

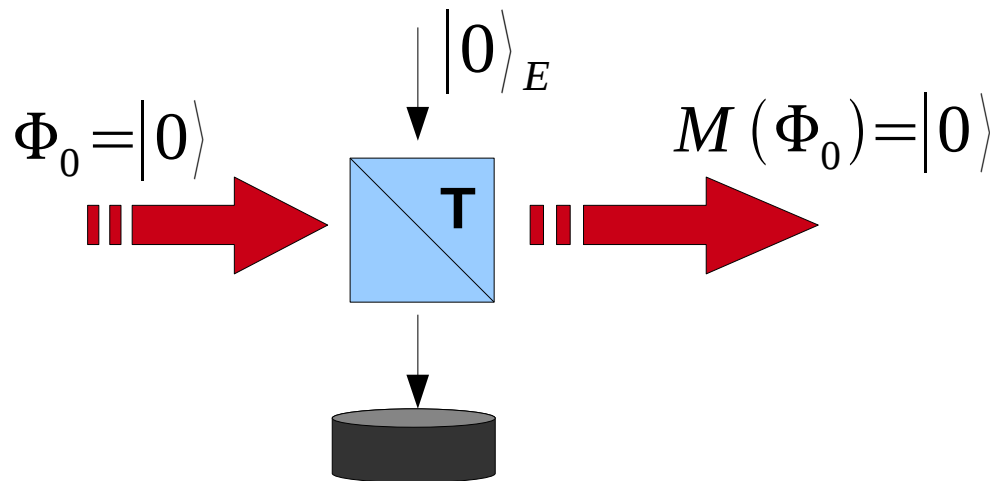
→ coherent states modulated with a Gaussian bivariate distribution do achieve the capacity (it is the *optimal encoding*)

... provided the **Gaussian minimum entropy conjecture** holds !

Gaussian Minimum Entropy Conjecture

$$\min_{\sigma} S(M(\sigma)) = S(M(|0\rangle\langle 0|))$$

- The same conjecture is made for the joint channel $M^{\times n}$
then, $C(M) \equiv \lim_{n \rightarrow \infty} \frac{1}{n} C^{(1)}(M^{\times n}) = C^{(1)}(M)$
- All papers (> 30) on the topic of Gaussian bosonic channels
rely on this widely admitted conjecture !!!
- Single exception: pure lossy channel (environment E in vacuum state)

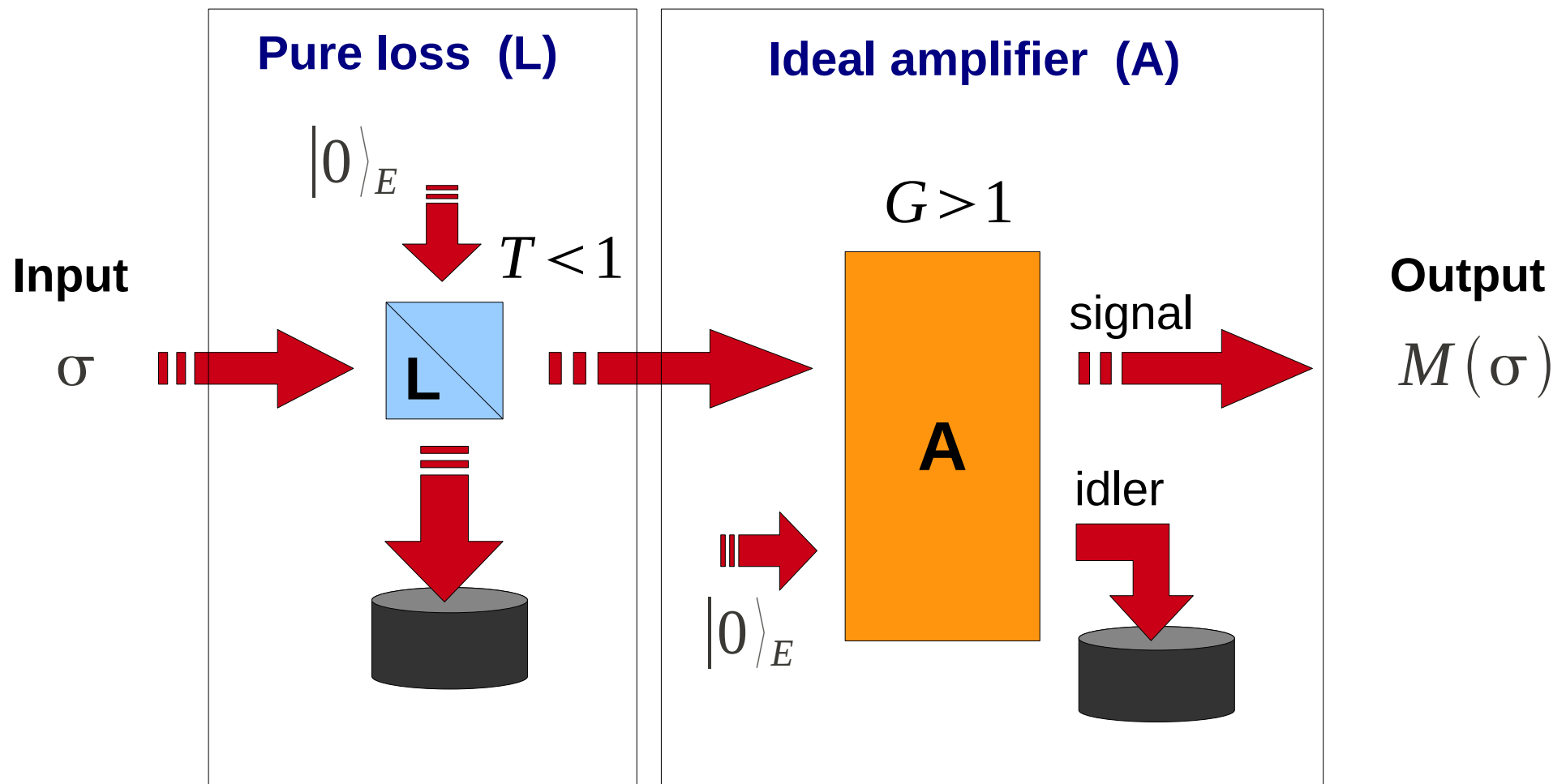


$$C^{(1)}(M) = g[\tau \nu]$$

V. Giovannetti *et al.*, PRL, 2004

Generic Decomposition of Phase-insensitive Channels

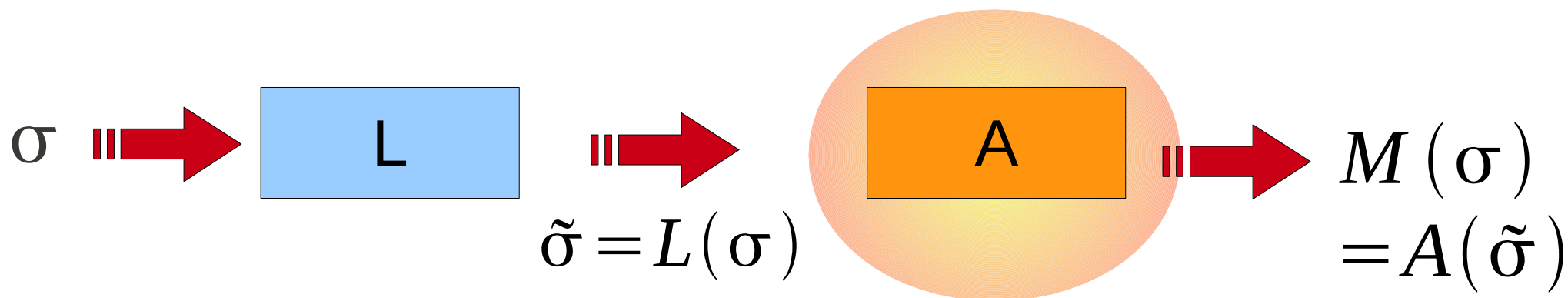
- $\tau = T G < 1$ lossy fiber with thermal noise
- $\tau = T G = 1$ classical Gaussian additive noise
- $\tau = T G > 1$ (non-ideal) noisy amplifier



Reduction of the Conjecture



Conjecture I $\min_{\sigma} S(M(\sigma)) = S(M(|0\rangle\langle 0|))$

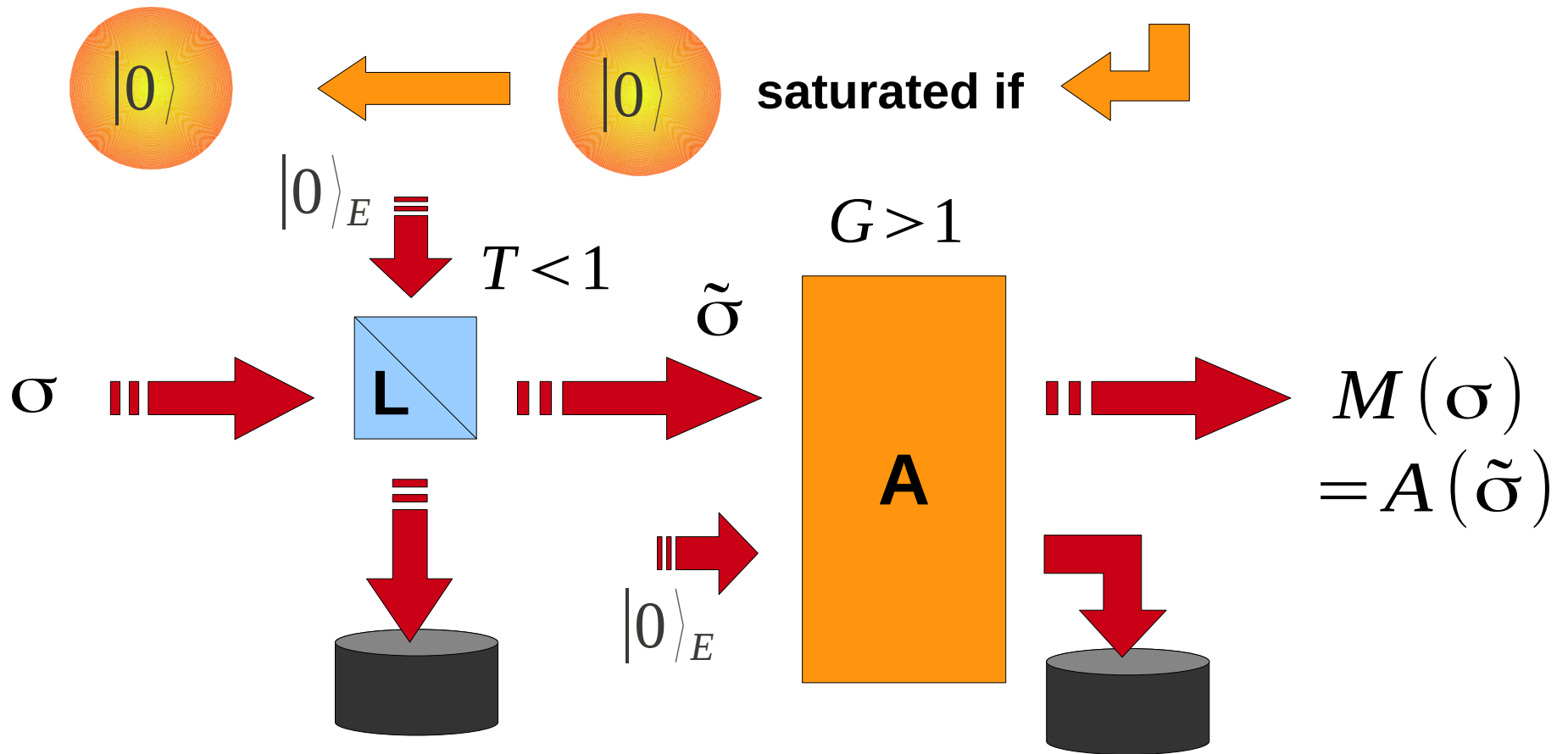


Conjecture II $\min_{\tilde{\sigma}} S(A(\tilde{\sigma})) = S(A(|0\rangle\langle 0|))$

Reduction to Ideal Amplifier

assume conjecture II holds

$$S(M(\sigma)) = S(A(\tilde{\sigma})) \geq S(A(|0\rangle\langle 0|))$$

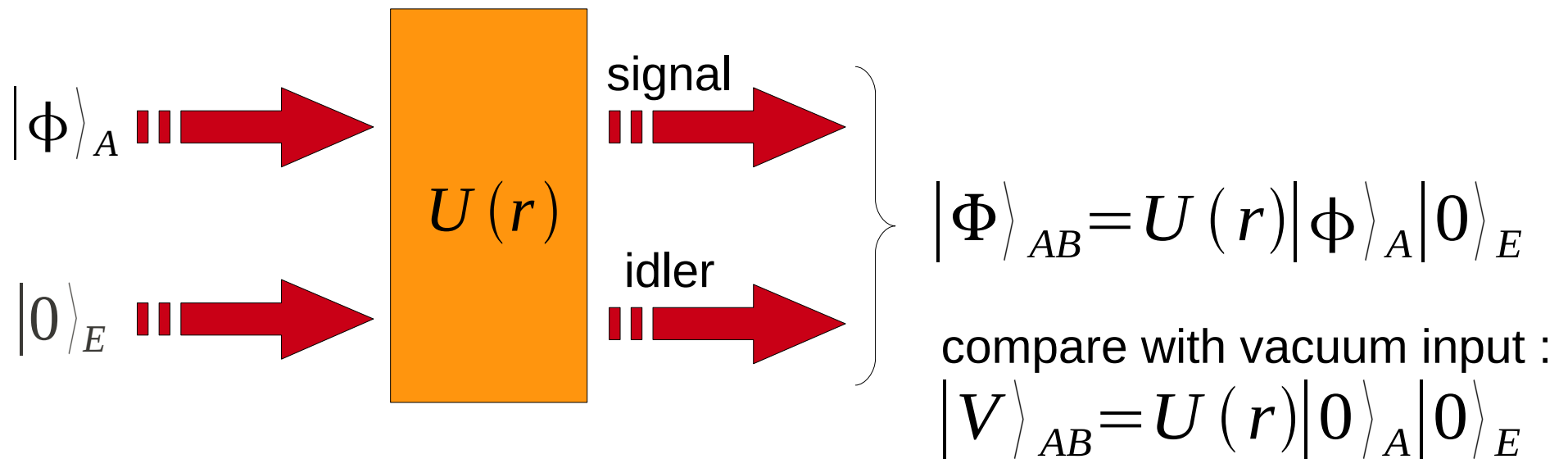


➡ it is (necessary and) sufficient to prove the reduced conjecture II

Gaussian Minimum Entanglement Conjecture

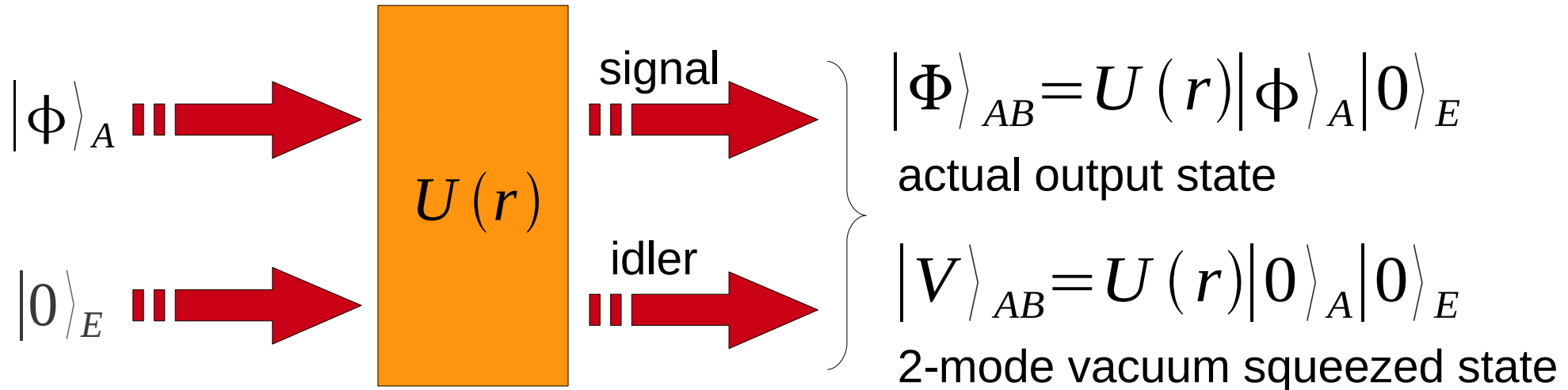


Conjecture I $\min_{\sigma} S(M(\sigma)) = S(M(|0\rangle\langle 0|))$



Conjecture II (bis) $\min_{\phi} E(|\Phi\rangle_{AB}) = E(|V\rangle_{AB})$

Proof for Gaussian vs non-Gaussian states



- easy to prove for **Gaussian** states

$$E(|\Phi_{Gauss}\rangle_{AB}) \geq E(|V\rangle_{AB})$$

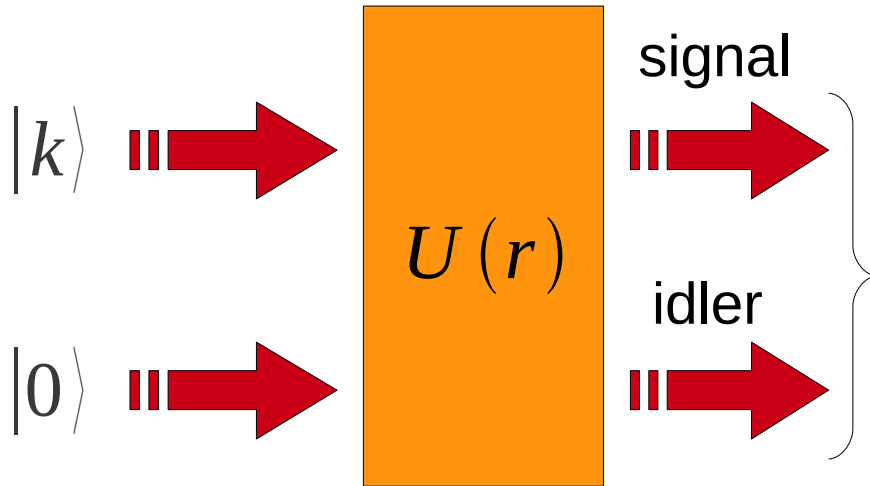
- expansion of **non-Gaussian** states in Fock basis

$$|\Phi\rangle_A = \sum_k c_k |k\rangle$$

← Fock states

... only an incomplete proof for Fock states !

Fock State Inputs

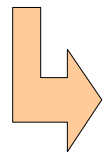


$$E(k) = H[P_n(k)] \quad \text{Shannon entropy}$$

$$P_n(k) = \frac{1}{\cosh^{2(k+1)} r} \binom{n+k}{n} \tanh^{2n} r$$

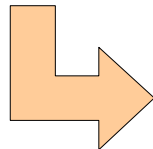
$$|\Psi_k\rangle = \sum_{n=0}^{\infty} \sqrt{P_n(k)} |n+k\rangle |n\rangle$$

Using Pascal identity $\binom{n+k+1}{n} = \binom{n+k}{n} + \binom{n+k}{n-1}$

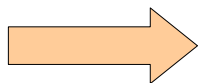


$$P_n(k+1) = (1-\lambda^2)P_n(k) + \lambda^2 P_{n-1}(k+1) \quad \text{with } \lambda = \tanh r$$

Concavity of entropy



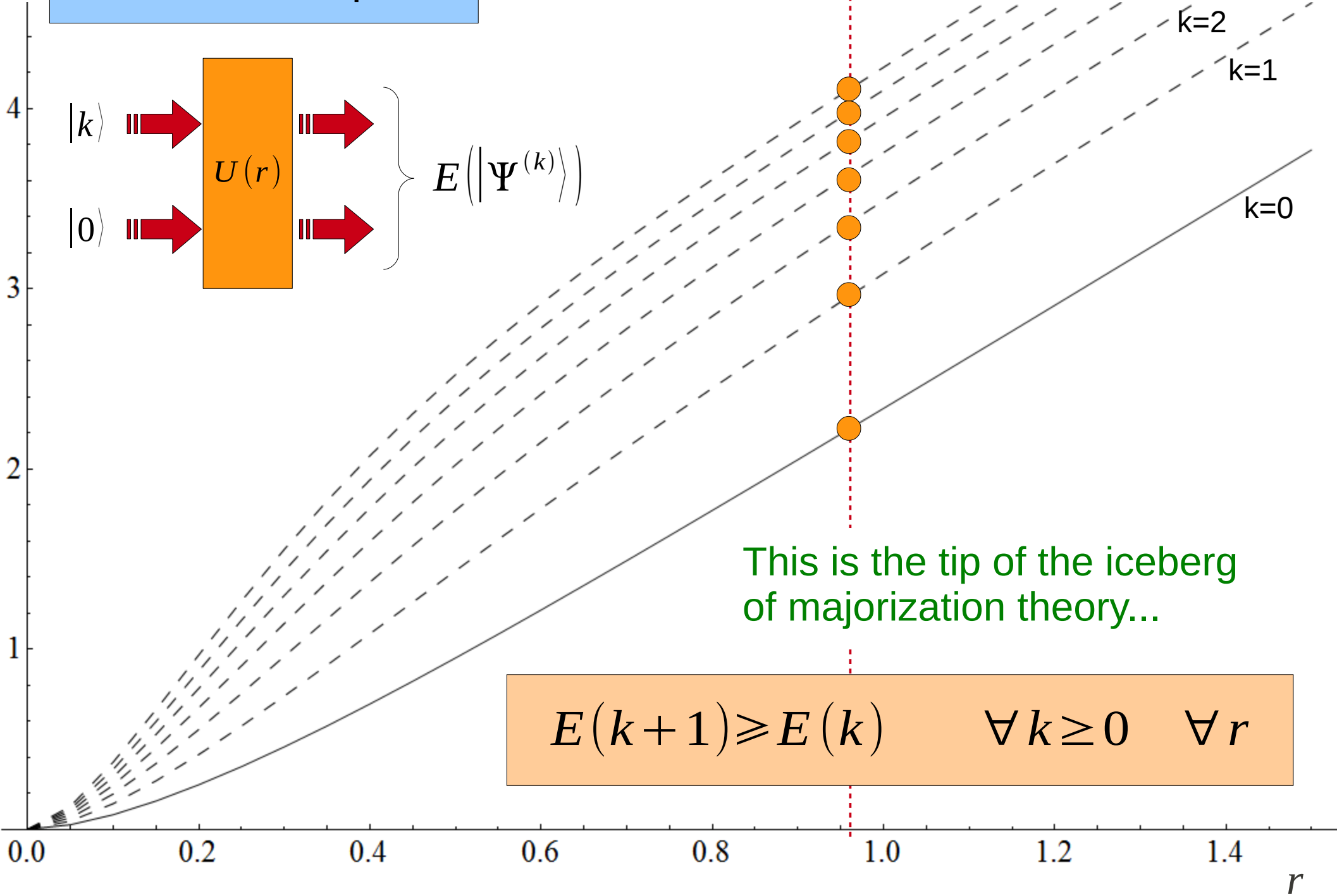
$$E(k+1) \geq (1-\lambda^2)E(k) + \lambda^2 E(k+1)$$



$$E(k+1) \geq E(k) \quad \forall k \geq 0$$

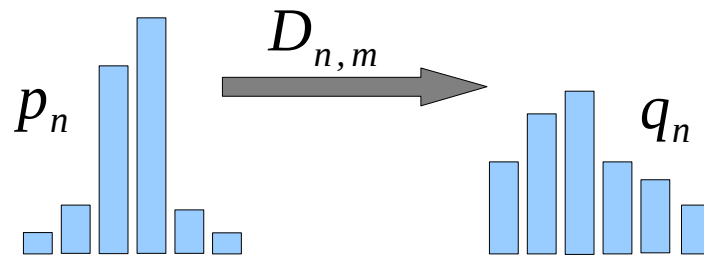
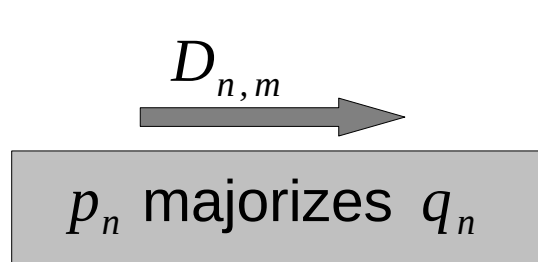
vacuum state beats
all other Fock states

Fock State Inputs



Majorization Theory

= (partial) order relation for probability distributions



with p_n, q_n probability distributions

if and only if

- p_n can be converted to q_n by applying a *random permutation*

$$q_n = \sum_m D_{n,m} p_m \quad D_{n,m} \text{ is doubly-stochastic matrix}$$

or ● $\sum_{n=0}^m p_n^\downarrow \geq \sum_{n=0}^m q_n^\downarrow \quad \forall m \geq 0$ (p is “*more peaked*” than q)

or ● $\sum_n h(p_n) \leq \sum_n h(q_n) \quad \forall h(x)$ concave function

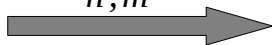
e.g. entropy: $h(x) = -x \log(x)$

→ $H(p_n) \leq H(q_n)$ entropy can only increase

Quantum case : Interconversion of pure bipartite states

M. Nielsen, G. Vidal, 2000

$D_{n,m}$



p_n majorizes q_n

p_n, q_n probability distributions

LOCC



$|\psi\rangle$ majorizes $|\phi\rangle$

$$|\psi\rangle = \sum_n \sqrt{p_n} |e_n\rangle |f_n\rangle \quad |\phi\rangle = \sum_n \sqrt{q_n} |e_n'\rangle |f_n'\rangle$$

with p_n majorizing q_n

- $|\phi\rangle$ can be converted to $|\psi\rangle$ by applying a deterministic LOCC
- $E(|\psi\rangle) \leq E(|\phi\rangle)$ **entanglement can only decrease**

Trick: $\rho_A = \text{tr}_B(|\psi\rangle\langle\psi|) = \sum_n p_n \underbrace{|e_n\rangle\langle e_n|}_{\text{orthonormal}}$ eigenbasis representation

$= \sum_n q_n \underbrace{|\xi_n\rangle\langle\xi_n|}_{\text{not orthonormal}}$ provided p_n majorizes q_n

Explicit conversion LOCC

LOCC



$|\psi\rangle$ majorizes $|\phi\rangle$

$$|\psi\rangle = \sum_n \sqrt{p_n} |e_n\rangle |f_n\rangle$$

$$|\psi\rangle = \sum_n \sqrt{q_n} |\xi_n\rangle |f_n''\rangle \quad \text{above trick (provided } p_n \text{ majorizes } q_n)$$

LOCC \uparrow \updownarrow U

$$|\phi\rangle = \sum_n \sqrt{q_n} |e_n'\rangle |f_n'\rangle$$

POVM: $A_m = \sum_n \omega^{nm} |\xi_n\rangle\langle e_n'|$ with $\omega = e^{i2\pi/d}$ and $\sum_m A_m^+ A_m = I$

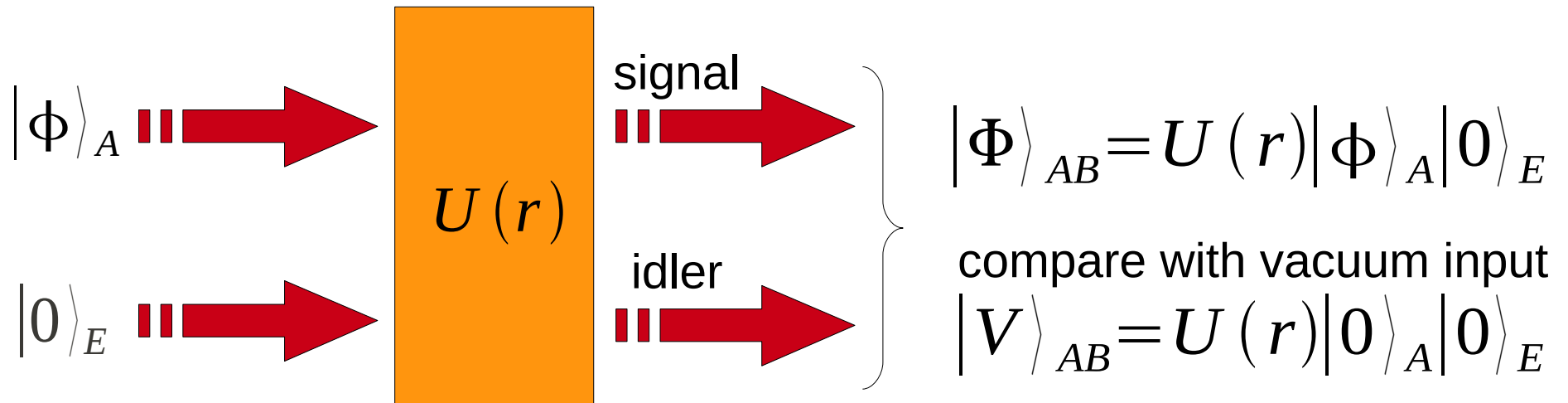
$$(A_m \times I) |\phi\rangle = \sum_n \sqrt{q_n} \omega^{nm} |\xi_n\rangle |f_n'\rangle \equiv |\phi_m\rangle \quad \text{depends on outcome } m$$

Conditional U: $B_m = \sum_n \omega^{-nm} |f_n''\rangle\langle f_n'|$ conditional on m

$$(I \times B_m) |\phi_m\rangle = \sum_n \sqrt{q_n} |\xi_n\rangle |f_n''\rangle \equiv |\psi\rangle$$

Gaussian Majorization Conjecture

For a given 2-mode squeezer (Bogoliubov transformation),
 the 2-mode vacuum squeezed state **majorizes** all other output states !!!



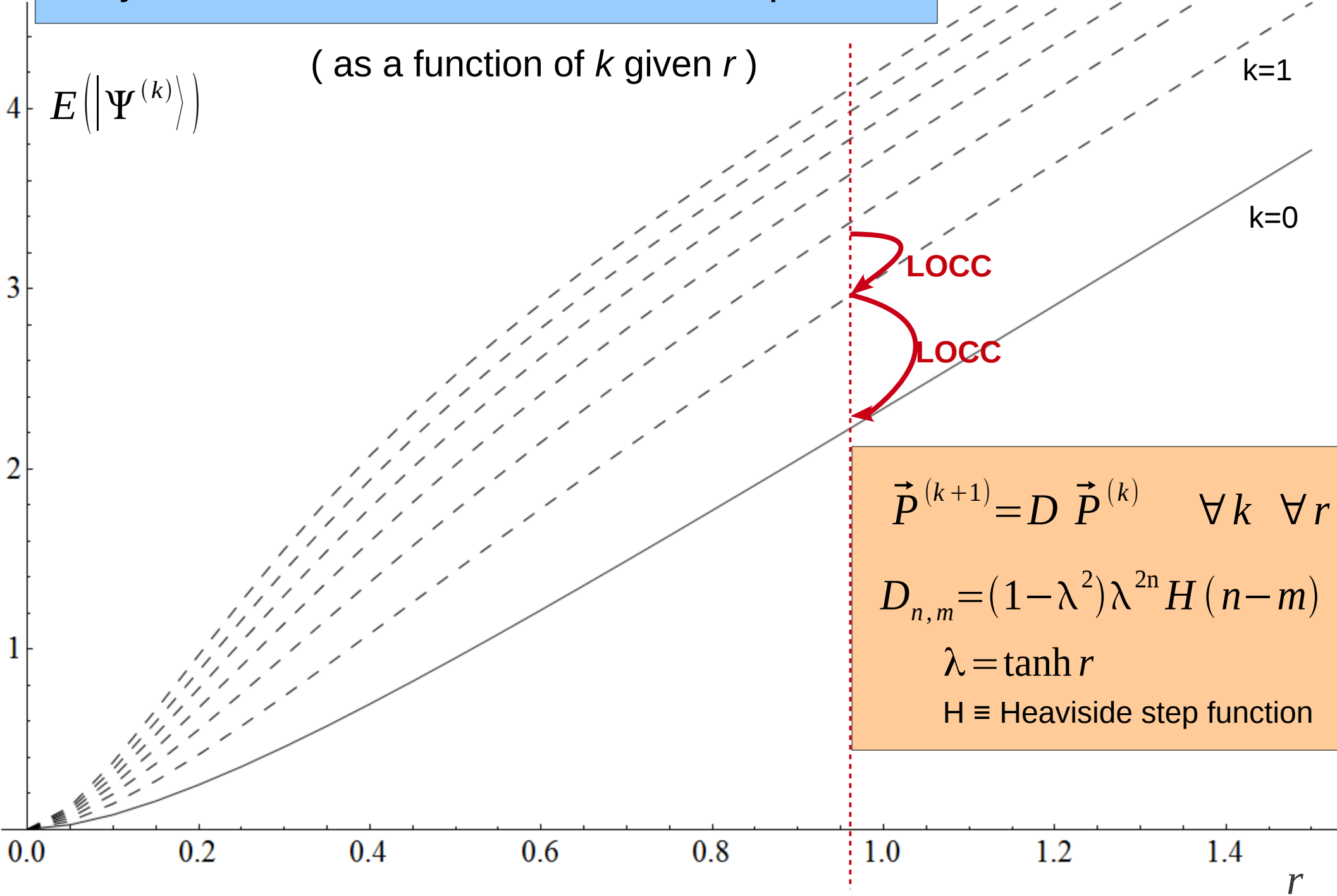
Conjecture III $|V\rangle_{AB}$ majorizes $|\Phi\rangle_{AB} \quad \forall |\Phi\rangle_A$

$$|V\rangle_{AB} \xleftarrow{\text{LOCC}} |\Phi\rangle_{AB} \quad \text{implying} \quad E(|V\rangle_{AB}) \leq E(|\Phi\rangle_{AB})$$

... stronger than minimum output entropy conjecture

... but perhaps easier to prove (?)

Majorization relations in a 2-mode squeezer



Explicit LOCC

$$|\Psi_k\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k)} |n+k\rangle|n\rangle$$

$$|\Psi_{k+1}\rangle = \sum_{n=0}^{\infty} \sqrt{p_n(k+1)} |n+k+1\rangle|n\rangle$$

Alice applies a POVM

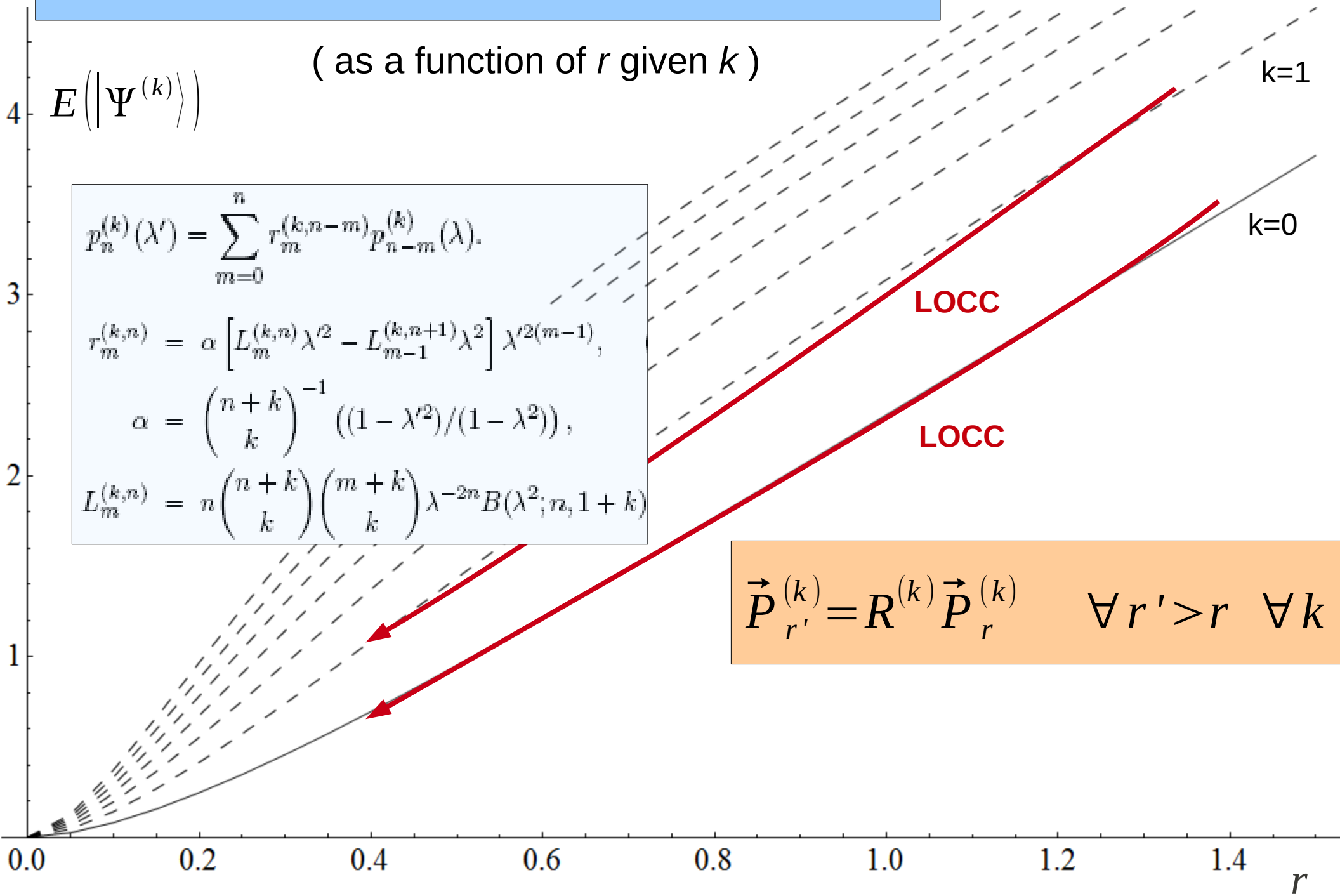
$$\left\{ \begin{aligned} A_{YES} &= \sum_{m=0}^{\infty} \sqrt{\frac{(1-\lambda^2)p_m(k)}{p_m(k+1)}} |m+k\rangle\langle m+k+1| \\ A_{NO} &= \sum_{m=0}^{\infty} \sqrt{\frac{\lambda^2 p_{m-1}(k+1)}{p_m(k+1)}} |m+k\rangle\langle m+k+1| \end{aligned} \right.$$

$$(A_{YES} \times 1) |\Psi_{k+1}\rangle = \sqrt{(1-\lambda^2)} \sum_{n=0}^{\infty} \sqrt{p_n(k)} |n+k\rangle|n\rangle = \sqrt{(1-\lambda^2)} |\Psi_k\rangle \quad \text{YES}$$

$$(A_{NO} \times 1) |\Psi_{k+1}\rangle = \sqrt{\lambda^2} \sum_{n=0}^{\infty} \sqrt{p_n(k+1)} |n+k+1\rangle|n+1\rangle \rightarrow \sqrt{\lambda^2} |\Psi_{k+1}\rangle \quad \text{NO}$$

If "NO" she communicates it to Bob who applies $U = \sum_{m=0}^{\infty} |m\rangle\langle m+1|$ and then they start a new round again

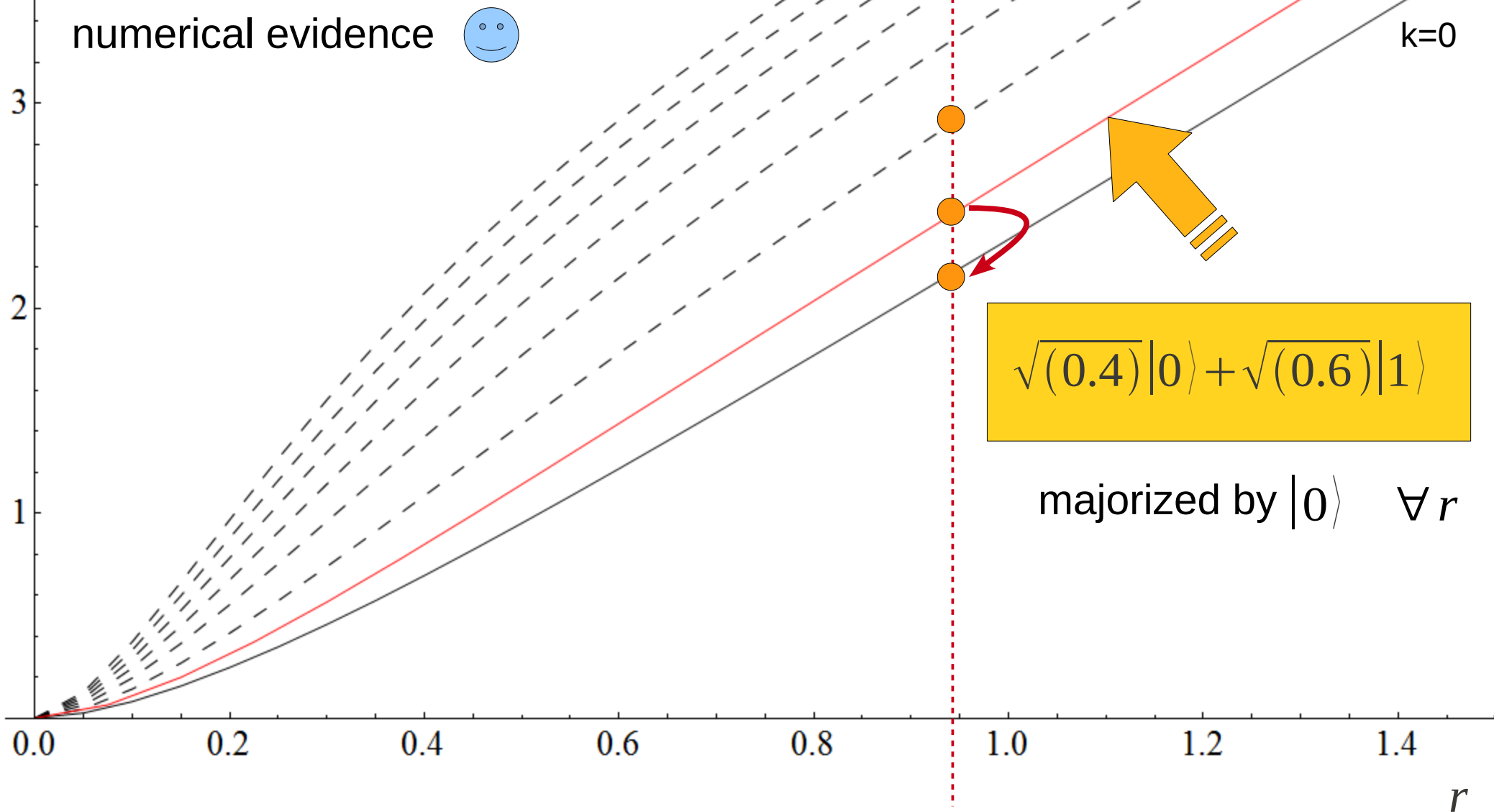
Majorization relations in a 2-mode squeezer



Arbitrary superposition of Fock states

$E(|\phi\rangle_{AB})$

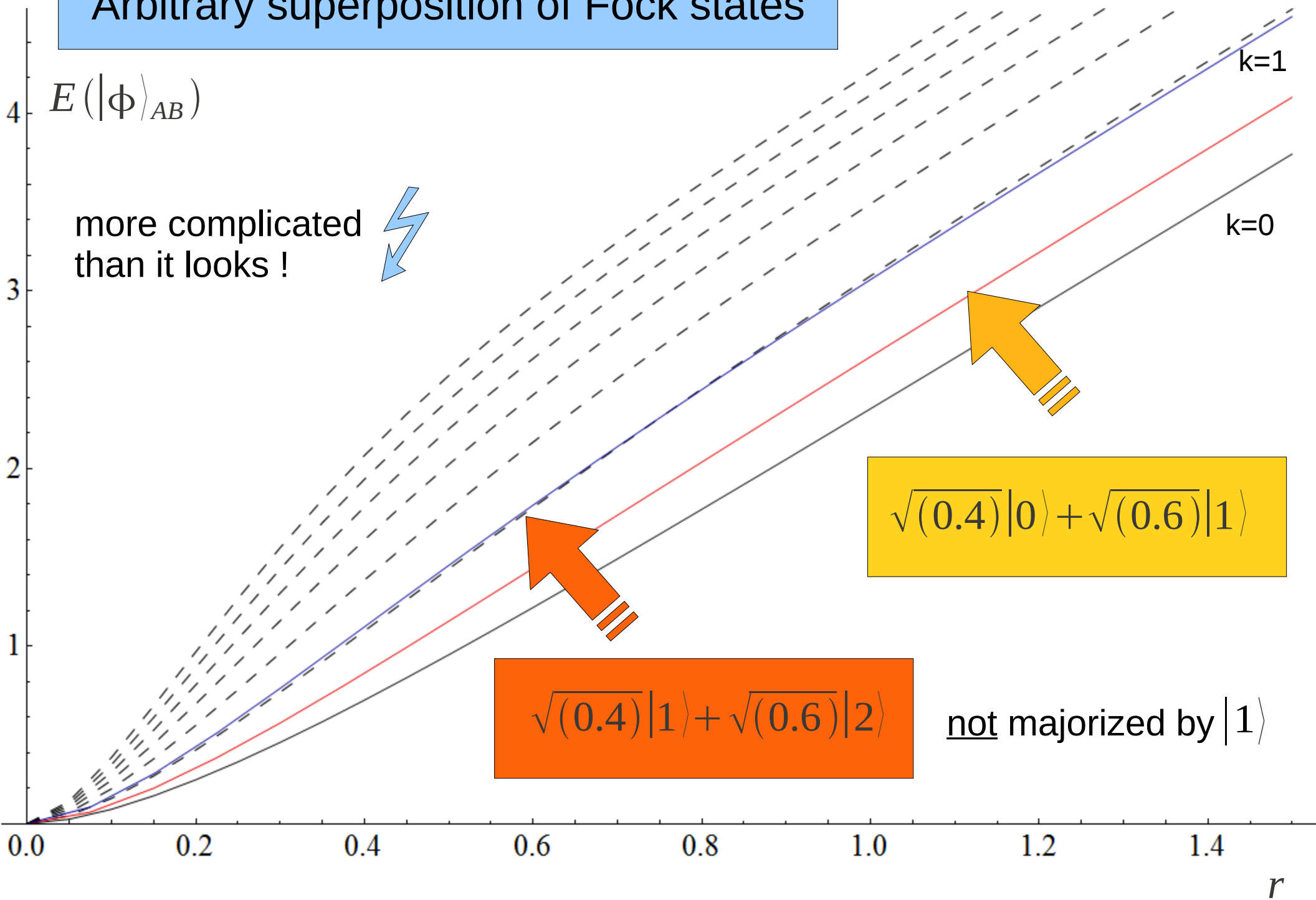
numerical evidence ☺



$$\sqrt{(0.4)}|0\rangle + \sqrt{(0.6)}|1\rangle$$

majorized by $|0\rangle \forall r$

Arbitrary superposition of Fock states



This is a fundamental problem

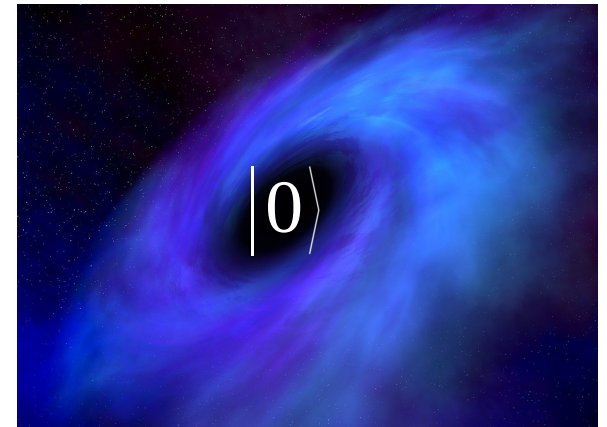
(even if you don't care
about quantum channels !)

Bogoliubov transformation $\hat{a}_i' = \sum_j (u_{ij} \hat{a}_j + v_{ij} \hat{a}_j^+)$ is everywhere
(e.g., quantum optics, superconductivity, Hawking radiation, Unruh effect,...)

... this conjecture may have implications there !

ALTERNATE TITLE: can we prove that ...

“Nothing is less than the vacuum” ?
(less random than the vacuum state)



Take-home message

“Nothing is less than the vacuum”

very plausible but not (yet) proven !

New approach to solve the “Minimum Output Entropy Conjecture”
for Gaussian bosonic channels

- Reduction to ideal amplifier channel
- Output entanglement of a two-mode squeezer
- Link with majorization theory (currently, Fock states only)

Numerical analysis for random input states
strongly suggests that (majorization) conjecture is true...

See: R. Garcia-Patron, C. Navarrete-Benlloch, S. Lloyd,
J. H. Shapiro & N. J. Cerf, arXiv:1111.1986 [quant-ph]